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ABSTRACT. An examination is presented of the quantity of information which can theoretically betransmitted over a quantum channel, together with an analysis of methods for realizing the potential possibilities of optical information transmission systems. The problem consists of determining the influence of the statistics of laser emission on the information capacity and of relating the statistics of photoelectrons to actual communications systems using a photo detector. It is judged that the most useful states are ones having the smallest dispersion of photons, which determines the fluctuation of the photoelectrons in the receiving apparatus. Before the introduction of information, the photon field has a minimum dispersion which increases after the addition of information.

It was noted in [1, 2] that the phenomena of coherence and fluctuation of laser radiation are directly related to various information effects which are of interest in constructing an actual optical range communications system. Of the problems arising during the investigation of this problem, we have separated the following two: the influence of the statistics of laser radiation on the information capacity of the radiation; the statistics of photoelectrons and actual communications systems containing photo detectors. These problems are closely interrelated since, on the one hand, the statistics of photoelectrons is determined by the statistics of photons, while on the other hand the information capacity of a photon field is realized in the final analysis using communications systems containing photo detectors. In other words, we will be interested in the quantity of information which can, in principle, be transmitted through a quantum communications channel, as well as the methods allowing these potential capacities to be realized. /229[1]

As we know, the quantum nature of laser radiation has forced the investigation of the influence of purely quantum effects on the process of information transmission. It has been shown [3-7] that the transition to quantum concepts cannot be achieved by simply transferring concepts from classical information theory. However, although certain positions from the new theory (such as the presence of quantum noise $h\nu\Delta\nu$ or the fixed level of entropy reference) cannot be called unexpected, others, generally speaking, require additional investigation. One of these is the problem of the influence of the process of

[1] Numbers in the margin indicate pagination in the foreign text.

information extraction on the total information properties of its carrier, particularly the properties of the light photon field. As we know, in classic wave optics, the process of measurement places no limitation on the state of the field being measured. If there is a signal whose frequency spectrum is concentrated in band width $\Delta\nu$, according to the classic conceptions, in principle two $\Delta\nu$ independent canonically coupled quantities can be measured with any required accuracy, for example the amplitudes of electrical and magnetic fields of the corresponding Fourier components. The position is otherwise in quantum theory. The uncertainty relationships leave only $\Delta\nu$ degrees of freedom, which naturally reduces the total information which can be extracted by measuring the states of the field. This source of reduction of the information capacity of a quantum channel in comparison to the classical concepts, however, is not the only such source. /230

We know that a quantum set located in the eigenstate of a certain $\hat{P}$-representation produces upon measurement a definite value only for those physical quantities whose operators commutate with operator $\hat{P}$. Unfortunately, actual communications systems cannot at the present time always record the required physical quantities. Thus, we know [7-10] that the coherent radiation of a single mode laser is an eigenstate in the representation of Glauber. On the other hand, all recording devices which currently exist measure energetic quantities, i.e. actually the number of photons arriving at the input of the instrument over a certain time τ. However, states with fixed numbers of photons are not the eigenstates in the Glauber representation. This means that in each given measurement, a number of values may be produced with various probabilities. Naturally, the formation of this additional uncertainty reduces the quantity of information which can be extracted by the recording device. Thus, after determining the maximum entropy of the photon field, according to [3-5],

$$S_{q.max.} = \bar{n} \ln\left(1 + \frac{1}{\bar{n}}\right) + \ln(1 + \bar{n}), \tag{1}$$

where $\bar{n}$ is the mean number of photons arriving per unit time and thereby satisfying the requirements related to the reduction in the number of degrees of freedom, we have still said nothing concerning the actual information capacity. In order to estimate this quantity, we must fix the quantity measured at the output of the channel, as well as the distribution of this quantity at the input of the device which introduces the information. -If this distribution is other than δ-shaped, entropy (1) can be arbitrarily divided into two parts: "useful" and "useless" entropy. The first portion can be used to introduce the information of interest to us and therefore determines the actual information capacity of the channel; the second results from statistical fluctuations in the physical quantity being measured at the input of the device introducing the information.

These considerations show that the relationship of the statistical properties of a field to its information properties are of considerable interest. Here we should note that in recent years a large number of works have appeared dedicated to the statistics of laser and thermal radiation and its recording (for example, [7-17]) in which essentially contradictory information is frequently encountered. This situation unfortunately occurs in defining the change in fluctuations in the number of photons with increased degrees of coherence of the radiation (see [11-13], also [17]). It seems natural to affirm that the /231 transition from a state with dispersion $\overline{\Delta n^2} = \bar{n}$ to a state with dispersion $\overline{\Delta n^2} = \bar{n}^2$, which corresponds to transition from the radiation of a source which is near harmonic to the radiation of an absolutely black body (where $\bar{n} \gg 1$) should be accompanied by a reduction in the "coherence," and not vice versa. Since we will be interested in the following in the information properties of the field, from this point of view the more preferable states are those with lower dispersion of the number of photons, since it is this dispersion which determines the fluctuation in the photoelectrons recorded by the receiving device. Thus, we can assume that before information is introduced, the field had minimum possible dispersion, and that the dispersion was increased after the information was introduced.

Information Capacity of a Photon Field

In the works of Stern [4, 5] and Gordon [3], it was shown that the information capacity I of a quantum communications channel is limited by the value of the maximum entropy

$$I < S_{q.max} = \bar{n}\ln\left(1 + \frac{1}{\bar{n}}\right) + \ln(1 + \bar{n}), \tag{2}$$

in which the distribution of photons corresponding to $S_{q.max}$ is exponential, i.e. the probability of the state with n photons is

$$P(n) = Ce^{-\sigma n}, \tag{3}$$

where

$$C = \frac{1}{1+\bar{n}}; \quad \sigma = \ln\left(1 + \frac{1}{\bar{n}}\right).$$

It is easy to show [5, 6] that distribution (3) corresponds to the equilibrium radiation of an absolutely black body; consequently, as in the classical case, in quantum theory the signal with maximum information capacity corresponds statistically to thermal noise. However, there is a difference in principle here from the classical situation, in that the possibility of realizing the entire value of entropy for transmission of messages, generally speaking, remains unproven. Furthermore, with existing methods of selection and processing of information in the optical frequency range, based on the usage of energy sensitive devices, complete utilization of entropy $S_{q.max}$ is impossible. This results from the fact that, as was mentioned above, the state of the photon field of a laser is not the eigenstate in the representation of occupation numbers and, therefore, the measurements of the energy sensitive receiver will include uncertainties related to the measurement process. Due to the additive nature of entropy, the maximum possible portion of $S_{q.max}$ which can be utilized /232 for the transmission of information is defined as the difference

$$\Delta S = S_{q.max} - S_0 \tag{4}$$

where S_0 is the entropy of the initial state of the field. Thus, the information capacity of the communications channel is limited by the relationship

$$I \leqslant \Delta S. \tag{5}$$

If in the initial state $S_0 = 0$, i.e. there are no fluctuations in the measured object, formula (5) is converted to the ordinary classical Shannon formula. It is this case which is analyzed in the classical theory where, in principle, realizations of states without fluctuations are always possible. If $S_0 \neq 0$, the signal, after the message has been introduced, will contain both "useful" information and a certain quantity of "useless" information, related to the random oscillations of the physical quantity at the input of the device which introduces the information perceived by the receiver as noise. Since we are analyzing energy sensitive recording devices here, we must find S_0 in the representation of occupation numbers. As a model, let us use the idealized case of a field in a single mode resonator without dissipation, since the presence of additional modes or absorption by the walls only increases the field fluctuation and consequently increases S_0.

As usual, we will seek the solution of the Schroedinger time equation in the form of a wave packet, which is transformed to the classical case at the limit of large numbers. We know that the Hamiltonian of the field in the resonator can be written in the form

$$H = \frac{1}{2}\left(\hat{p}^2 + \omega^2\hat{q}^2\right) = \hbar\omega\left(\hat{n} + \frac{1}{2}\right), \tag{6}$$

where

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\left(\hat{a} + \hat{a}^{+}\right);$$
$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}\left(\hat{a}^{+} - \hat{a}\right);$$
$$\hat{n} = \hat{a}^{+}\hat{a}.$$

Operators $\hat{a}$ and $\hat{a}^{+}$ are the operators of destruction and generation of photons respectively; $\hat{n}$ is the operator of the number of particles. eigen functions of of Hamiltonian (6) are the solution to the Schroedinger equation

$$H\Psi_n = E_n\Psi_n \tag{7}$$

for the harmonic oscillator, which can be written in the following form [18]:

$$\Psi_n(q) = N_n e^{-\frac{\lambda^2 q^2}{2}} H_n(\lambda q), \tag{8}$$

where /233

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega;$$
$$N_n = \left(\frac{\lambda}{\pi^{1/2} 2^n n!}\right)^{1/2};$$
$$\lambda = \sqrt{\frac{\omega}{\hbar}};$$

H_n is the Hermit n-th order polynomial. It is not difficult to see that in the stationary state, the following equalities are fulfilled:

$$\langle q \rangle = \langle p \rangle = 0;$$
$$\Delta q \Delta p = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{9}$$

Let us now find the solution to the Schroedinger time equation

$$i\hbar \frac{\partial \Psi(q, t)}{\partial t} = H\Psi(q, t), \tag{10}$$

which can be written in the form of an expansion with respect to the stationary wave functions

$$\Psi(q, t) = \sum_n A_n \Psi_n(q) e^{-\frac{iE_n t}{\hbar}}. \tag{11}$$

We require that at moment t = 0 the fluctuations of the field be minimal, i.e.

$$\Delta p \Delta q = \frac{\hbar}{2}, \tag{12}$$

and, furthermore, the center of the wave packet be displaced in the positive direction of p and q such that

$$\begin{aligned} \langle q \rangle &= q(0); \\ \langle p \rangle &= p(0). \end{aligned} \tag{13}$$

From this we produce

$$\Psi(q, 0) = \sum_n A_n \Psi_n = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left\{-\frac{\omega}{2\hbar}[q - q(0)]^2 - \frac{ip(0)}{\hbar} q\right\}. \tag{14}$$

It is not difficult to see that in this case relationships (13) are fulfilled. Equation (14) allows us to find the coefficients for the expansion of A_n. This calculation is performed, for example, in [18], producing the following:

$$|A_n|^2 = \frac{\bar{n}^n e^{-\bar{n}}}{n!}; \tag{15}$$

$$\bar{n} = \frac{E_{\text{кл}}}{\hbar\omega} = \frac{1/2\,[p^2(0) + \omega^2 q^2(0)]}{\hbar\omega}. \tag{16}$$

The value of $|A_n|^2$ is the probability of the state with fixed number of photons n. This allows us to calculate the entropy related to fluctuations at the input of the device which introduces the information: /234

$$S_0 = -\sum_{n=0}^{\infty} |A_n|^2 \ln |A_n|^2 = -\sum_{n=0}^{\infty} \frac{\bar{n}^n e^{-\bar{n}}}{n!} \ln \left[\frac{\bar{n}^n e^{-\bar{n}}}{n!} \right]. \tag{17}$$

Thus, the minimum entropy of the field in this case is determined by the entropy of the Poisson distribution. With large $\bar{n}$, the Poisson distribution can be replaced with sufficient accuracy by the Gaussian distribution, i.e.

$$S_0 \approx \frac{1}{2} \ln (2\pi e\bar{n}). \tag{18}$$

Expression (1) for the maximum entropy of the signal with average power $\bar{n}\hbar\omega$ in turn can be rewritten for large $\bar{n}$ in the form

$$S_{\text{q.max.}} \approx \ln (e\bar{n}), \tag{19}$$

from which

$$I \leqslant \ln (e\bar{n}) - \frac{1}{2} \ln (2\pi e\bar{n}) = \frac{1}{2} \ln (e\bar{n}) - \frac{1}{2} \ln 2\pi = \frac{1}{2} S_{\text{q.max.}} - \frac{1}{2} \ln 2\pi. \tag{20}$$

Consequently, only approximately one half of the maximum entropy of the photon field can be used for the transmission of information. The remaining portion of the entropy, resulting from the peculiarities of quantum theory related to the possibility of measuring a definite physical quantity in a fixed representation, cannot be separated by energy sensitive recording devices, and is not available for the transfer of information.

Information Losses During Photo Detection

The result produced in the preceding section was discovered in the first works of Stern [4, 5], which were dedicated to quantum theory of information for the particular case of an ideal quantum amplifier. Similar relationships were produced by the authors of the present article for the case of a multichannel communications system with frequency division of channels [19], where the process of separating the flux of photons into channels also involves a loss of

approximately one half of all the information. In the following we will find the quantity of information separated by a photo detector and show that with sufficiently high quantum effectiveness of the photo detector its information effectiveness is also determined by formula (20) (this was partially done in [23]). In this work, we will base ourselves on the relationships produced by Mandel [12, 13, 15, 16], relating the statistical distribution of photons at the input of a photo detector to the distribution of the output photoelectrons. /235
If at the input of the photosensitive surface, the intensity of the incident light changes in the time interval [t,t + T] according to the rule I(t'), the probability of recording precisely k photoelectrons in this time interval is determined, according to Mandel, from the formula

$$P(k, t, T) = \frac{1}{k!}[\alpha U(t, T)]^k \exp[-\alpha U(t, T)], \tag{21}$$

where

$$U(t, T) = \int_t^{t+T} I(t')\,dt'; \tag{22}$$

α is the quantum effectiveness of the photosensitive surface.

Distribution (21) describes the recording of photoelectrons for a definite realization of the input field. Actually, this quantity cannot be observed; the observed quantity is the distribution of photoelectrons averaged over a rather long time interval. If the random process describing the field at the input of the photo detector has the properties of stability and ergodicity, it can be assumed in most cases of interest to practice that this averaging is independent of the selection of a reading time t and corresponds to the average for the set. Thus, if the value of U at the input can be considered continuous, we produce finally

$$P(k, T) = \int_0^\infty \frac{1}{k!} e^{-\alpha U} (\alpha U)^k P(U)\,dU, \tag{23}$$

where P(U) is the distribution of the classical field intensity at the input of the photo detector; P(k,T) is the probability of recording k photoelectrons during the integration time T.

If the number of photons at the input is not great, intensity U has an essentially discrete distribution and, consequently, the distribution of the classical intensity P(U) should be replaced by the distribution of the occupation numbers of the photons P(n), where $n = U/\hbar\omega$, and integration in (23) by the corresponding summation:

$$P(k, T) = \sum_{n=0}^{\infty} \frac{1}{k!} e^{-\alpha n} (\alpha n)^k P(n). \tag{24}$$

Relationships (23) and (24) are of great significance for the investigaticn of optical information transmission systems, since practically all such systems contain a photo detector. Below, on the basis of these formulas, we shall produce the quantity of information I(k,n) at the output of the photo detector and show the relationship of I(k,n) with the value of I in the preceding section. For this, we consider that the quantity of information I(k,n) can be written in the form /236

$$I(k, n) = S(k) - S(k/n), \tag{25}$$

where S(k) is the entropy of the output distribution of the photoelectrons;

$$S(k/n) = -\sum_{n=0}^{\infty} P(n) \sum_{k=0}^{\infty} P_n(k) \ln P_n(k) - \tag{26}$$

is its conditional entropy. Here $P_n(k)$ is the probability that when n photons arrive at the input of a detector, precisely k photoelectrons will appear at its output. From this definition and formula (24), it is easy to produce the explicit form of $P_n(k)$:

$$P_n(k) = \frac{(\alpha n)^k}{k!} e^{-\alpha n}. \tag{27}$$

Suppose now the distribution of photons at the input is fixed by expression (3), so that the entropy of the input signal is maximal and equal to (1). In order to calculate S(k) and S(k/n), we find further

$$P(k) = C \sum_{n=0}^{\infty} \frac{1}{k!} e^{-\alpha n} (\alpha n)^k e^{-\sigma n} \approx C' e^{-\sigma' k}, \tag{28}$$

where

$$C' = \frac{C}{\alpha+\sigma} = \frac{1}{1+\bar{n}\alpha};$$
$$\sigma' = \ln\left(1+\frac{\sigma}{\alpha}\right) \approx \ln\left(1+\frac{1}{\alpha\bar{n}}\right). \tag{29}$$

Using the expressions produced, we find

$$S(k) = -\sum_{k=0}^{\infty} P(k)\ln P(k) \approx \ln(e\bar{n}\alpha). \tag{30}$$

Unfortunately, we cannot yet calculate S(k/n). However, we can find approximate estimates for it, as was done in [4]. Replacing the entropy of the Poisson distribution by the entropy of the Gaussian distribution, and replacing the sum with respe t to n in (26) with integration, we produce

$$S(k/n) = \frac{1}{2}\ln(2\pi e\bar{n}\alpha) - B, \tag{31}$$

where

$$-1/2 \leqslant B \leqslant 1/2. \tag{32}$$

Thus, we produce the following final expression for the quantity of information which can be extracted by a photo detector:

$$I(k,n) = \frac{1}{2}\ln(e\bar{n}) - \frac{1}{2}\ln\frac{2\pi}{\alpha} + B. \tag{33}$$

With sufficiently high effectiveness of the photo detector, this latter expression corresponds with an accuracy to a nonessential component with (20) /237 and, consequently, only one half of the entropy of the input distribution is information entropy. However, if $\alpha \ll 1$, the loss of information may be even greater. For example, where $\alpha = 0.01$; $\bar{n} = 10^4$, we have $I(k,n) \approx 0.2\ S(n)$. Let us emphasize in conclusion that the correspondence of formulas (20) and (33), concluded on the basis of quite different moduli, results from the specific quantum features of the information carrier, the photon field, as they appear in different situations.

The Photo Detector in Actual Communications Systems

Let us produce a certain relationship characterizing the operation of the photo detector in an actual communications system. In order to do this, let us analyze the receiver shown on Figure 1. We can ask the question how effective is the usage of the quantum optical amplifier in this receiver for preliminary amplification of the signal when operating with a photo detector. Unfortunately, there is not yet any sufficiently universal criterion to answer this question. The estimates used in radio engineering, based on the signal/noise ratio, cannot be used here, particularly with low level input signals, when only a few quanta are recorded during the time of integration of the photo detector. A more acceptable quantity is that suggested by Steinberg [20], which can be written in slightly altered form as follows:

$$\gamma = \frac{\left[\bar{k}_{s+n} - \sqrt{\overline{\Delta k^2_{s+n}}}\right] - \left[\bar{k}_n + \sqrt{\overline{\Delta k^2_n}}\right]}{\bar{k}_{s+n}}, \tag{34}$$

where $\bar{k}_n$, $\overline{\Delta k^2_n}$ and $\bar{k}_{s+n}$, $\overline{\Delta k^2_{s+n}}$ are the mean number and dispersion of electrons at the output of the photo detector when noise alone is received and when noise and signal are superimposed respectively. The sense of γ is clear from Figure 2. The greater γ, the easier it is to distinguish between noise alone and the superposition of signal plus noise and, therefore, the higher the probability of detecting the signal. It can therefore be affirmed that the usage of the quantum optical amplifier is expedient when the value of γ, called the separation parameter, increases in comparison to a receiver without the amplifier.

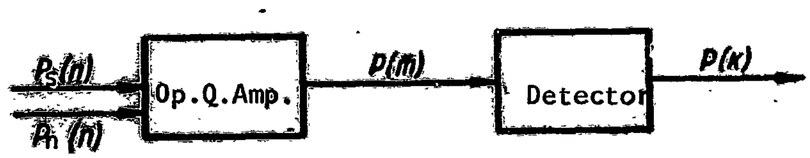

Figure 1. Passage of Signal and Noise Through Simple Optical Range Receiver

In order to determine γ, we must calculate the mean value and dispersion of the distribution of photoelectrons at the output of the detector with and without the optical quantum amplifier. In order to do this, we use formula (24), as well as an expression produced by Shimoda et al. [22] which determines the distribution of photons P(m) at the output of the OQA: /238

$$P(m) = \sum P(n)\, q_n(m), \tag{35}$$

where

$$q_n(m) = \begin{cases} \binom{-L-n}{m-n}\left(\frac{1}{G}\right)^{L+n}\left(\frac{1}{G}-1\right)^{m-n} & \text{where } m > n, \\ 0 & \text{where } m < n; \end{cases} \tag{36}$$

$P(n)$ is the input distribution of photons; $q_n(m)$ is the negative binomial distribution; L is the internal noise of the amplifier; G is its gain. It is easy to see from (24) that the values of the mean number of photoelectrons and their dispersion at the output of the detector are related to the corresponding values of distribution of photons at its input by the following relationships:

$$\begin{gathered} \bar{k} = \alpha \bar{m}; \\ \Delta \overline{k^2} = \alpha^2 \Delta \overline{m^2} + \alpha \bar{m}. \end{gathered} \tag{37}$$

In the same way, the mean values and dispersion at the output of the OQA are determined through the input quantities using the formulas:

$$\begin{gathered} \bar{m} = G\bar{n} + L(G-1); \\ \Delta \overline{m^2} = G^2 \Delta \overline{n^2} + G(G-1)(\bar{n} + L). \end{gathered} \tag{38}$$

Finally, we must consider that during the additive combination of two independent, random quantities, their mean values and dispersions follow the relationships:

$$\begin{gathered} \overline{(x+y)} = \bar{x} + \bar{y}; \\ \Delta \overline{(x+y)^2} = \Delta \overline{x^2} + \Delta \overline{y^2}. \end{gathered} \tag{39}$$

Considering (37)-(39), it is easy to produce the following expression for the separation parameter:

$$\gamma(\alpha, G, L) = \frac{1}{\bar{n}_s + \bar{n}_n + L\left(1 - \frac{1}{G}\right)} \left\{ \bar{n}_s - \right.$$ /239

$$- \sqrt{\Delta\bar{n}^2_s + \Delta\bar{n}^2_n + \left(1 - \frac{1}{G}\right)(n_s + \bar{n}_n + L) + \frac{1}{\alpha G}\left[\bar{n}_s + \bar{n}_n + L\left(1 - \frac{1}{G}\right)\right]} -$$

$$\left. - \sqrt{\Delta\bar{n}^2_n + \left(1 - \frac{1}{G}\right)(\bar{n}_n + L) + \frac{1}{\alpha G}\left[\bar{n}_n + L\left(1 - \frac{1}{G}\right)\right]} \right\}, \tag{40}$$

in which the system without the amplifier corresponds to the quantity $\gamma(\alpha, 1)$ where $G = 1$. The estimation of the effectiveness of the work of the amplifier is performed by determining the difference

$$\Delta\gamma = \gamma(\alpha, G, L) - \gamma(\alpha, 1), \tag{41}$$

which should be positive if the usage of the amplifier is to yield any improvement. In many cases, rather simple formulas can be produced for the value of $\Delta\gamma$. For example, if the amplifier is near ideal, i.e. $L \ll \bar{N}_n$, $\bar{N}_s$, it is not difficult to see that

$$\Delta\gamma \approx \frac{1}{\bar{n}_s + \bar{n}_n}\left[\sqrt{\Delta\bar{n}^2_s + \Delta\bar{n}^2_n + \frac{1}{\alpha}(\bar{n}_s + \bar{n}_n)} - \right.$$

$$- \sqrt{\Delta\bar{n}^2_s + \Delta\bar{n}^2_n + \left(1 - \frac{1}{G} + \frac{1}{\alpha G}\right)(\bar{n}_s + \bar{n}_n)} + \tag{42}$$

$$\left. + \sqrt{\Delta\bar{n}^2_n + \frac{1}{2}\bar{n}_n} - \sqrt{\Delta\bar{n}^2_n + \left(1 - \frac{1}{G} + \frac{1}{\alpha G}\right)\bar{n}_n} \right] > 0$$

for all $G > 1$. If we assume further that the distribution of the signal and noise at the input is Poisson, i.e. $\Delta\bar{n}^2_n = \bar{n}_n$, $\Delta\bar{n}^2_s = \bar{n}_s$, we can produce a particularly simple expression for the gain realized

$$\Delta\gamma = \sqrt{1+\frac{1}{\alpha}}\left[1-\sqrt{1-\frac{1-\alpha}{1+\alpha}\left(1-\frac{1}{G}\right)}\right]\times$$
$$\times\frac{\sqrt{\bar{n}_s+\bar{n}_n}+\sqrt{\bar{n}_n}}{\bar{n}_s+\bar{n}_n}, \tag{43}$$

from which it follows that the usage of an ideal amplifier is more effective, the less the quantum sensitivity of the detector and the mean power of signal and noise at the input of the receiver and the greater the gain G. It is interesting to note that at the limit of very large G, the separation parameter approaches the following quantity

$$\gamma_{q.max.}=\frac{\bar{n}_s}{\bar{n}_s+\bar{n}_n}-\sqrt{2}\,\frac{\sqrt{\bar{n}_s+\bar{n}_n}+\sqrt{\bar{n}_n}}{\bar{n}_s+\bar{n}_n}, \tag{44}$$

which determines the separation of the signal and noise in the system without amplifier, but with quantum effectiveness of the photo detector equal to 1.

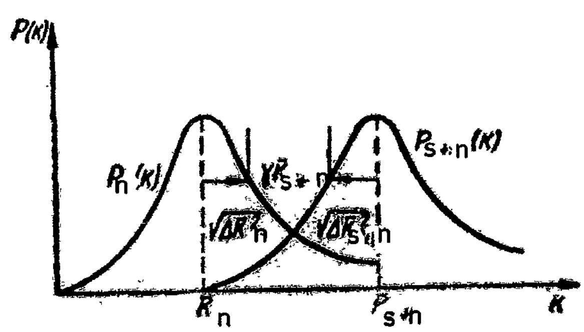

Figure 2. Explanation of Physical Sense of Separation Parameter γ

In the case of an amplifier with very high internal noise ($L \gg \bar{n}_s, \bar{n}_n$), assuming once more the input distribution of the signal and noise to be a Poisson distribution, we produce /240

$$\Delta\gamma\approx\sqrt{1+\frac{1}{\alpha}}\,\frac{\sqrt{\bar{n}_s+\bar{n}_n}+\sqrt{\bar{n}_n}}{\bar{n}_s+\bar{n}_n}-\frac{\bar{n}_s}{\bar{n}_s+\bar{n}_n}-$$
$$-2\sqrt{1+\frac{1}{\alpha G}}\,\frac{1}{\sqrt{L}}, \tag{45}$$

from which, assuming $\bar{n}_s \gg \bar{n}_n$, $\bar{n}_s > 1$, we find that the amplifier yields an additional gain in the case when

$$\alpha < \frac{1}{n_0}. \tag{46}$$

It must be noted that the difference in the results produced from the results of Steinberg, who found that the usage of an amplifier which was near ideal yields an increase in gain only under the condition $\alpha < 1/2$, can be explained by the difference in the description of the photo detector used. The usage of a binomial distribution in place of (24), which was done in [20], changes the value of the dispersion of the photoelectrons, and in the final analysis changes the conditions applied on the quantum effectiveness of the detector. Some additional difference in the quantitative relationships also results from differences in definition of the separation parameter γ.

The description of the optical range receiver device presented above, based on the introduction of the separation parameter γ, although it does have a number of advantages when compared with the description made using the signal to noise ratio, is still not completely satisfactory. For example, the possibility of existence of $\gamma < 0$, generally speaking, makes analysis of the operation of the device more difficult. For certain distributions of input signal and noise (for example, the exponential distribution) the introduction of γ has no clear sense at all, since these distributions have no maximum near their mean value. Also, in many cases an increase in input noise may cause an increase in the separation parameter, not a decrease, as would be expected. This defect does not arise if the separation parameter γ is determined according to Steinberg. However, in this case, γ does not allow of simple physical interpretation as in our case. These difficulties force us to search for a more acceptable criterion to be used in evaluating photo receptor operation in communications systems. In many cases, it is more convenient to begin with direct calculation of the mean error at the output of the receiver, than minimize it subsequently. Let us analyze one system allowing this type of analysis, namely a binary system with passive pause. This type of system is of considerable interest for operation in the optical range, and has been investigated in the literature repeatedly [11, 21, 24-25], however most of this analysis has only been qualitative, which in some cases has led to factual errors. We present below a rather general description of the receiving device of a binary optical range communications system and its optimization. /241

Let us analyze the system shown on Figure 3. The passage of the input signal through the receiver channel causes a change not only in the mean signal power, but in the type of distribution as well, in correspondence with formulas (24 and (35). The output device, which has a certain fixed threshold, is to differentiate the cases "signal plus noise" and "noise," developing the signals which will be represented in the following by "1" and "0" respectively. The effectiveness of this process depends both on the value of the threshold, and on the distribution of probability of appearance of a fixed number of photoelectrons at the output of the detector. If a certain number of photoelectrons arriving at the detector during the integration time of the detector is selected as the threshold, we can determine the probability of error as follows:

$$P_1 = \sum_{k=N+1}^{\infty} P_n(k), \quad P_2 = \sum_{k=0}^{N} P_{s+n}(k). \tag{47}$$

Here P_1 is the probability that the symbol "1" will be received in place of "0"; P_2 is the probability that "0" will be received in place of "1"; $P_n(k)$ and $P_{s+n}(k)$ are the distributions of photoelectrons resulting from the action of input noise and the superposition of signal plus noise respectively. Assuming that the mean frequency of appearance of the symbols "1" and "0" is identical, the overall probability of error at the output of the receiver can be represented in the form

$$P_{er} = \frac{1}{2}(P_1 + P_2) = \frac{1}{2}\left\{1 + \sum_{k=0}^{N} [P_{s+n}(k) - P_n(k)]\right\} \tag{48}$$

or, using (24) and (25),

$$2P_{er} = 1 + \sum_{k=0}^{N} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_n(m) p_m(k) [P_{s+n}(n) - P_n(n)], \tag{49}$$

where

$$P_{s+n}(n) = \sum_{l=0}^{n} P_s(l) P_n(n-l) \tag{50}$$

is the distribution of the sum of the two independent random events. Now substituting $q_n(m)$ and $p_m(k)$, we produce /242

$$2P_{er} = 1 +$$

$$+ \sum_{k=0}^{N} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \binom{-L-n}{m-n} \left(\frac{1}{G}\right)^{L+n} \left(\frac{1}{G} - 1\right)^{m-n} \frac{(\alpha m)^k}{k!} e^{-\alpha m} [P_{s+n}(n) -$$

$$- P_n(n)] = \bar{1} + \sum_{k=0}^{N} \sum_{n=0}^{\infty} \frac{\alpha^k}{k!} (-1)^k \left(\frac{1}{G}\right)^{L+n} [P_{s+n}(n) - P_n(n)] \times$$

$$\times \frac{d^k}{d\alpha^k} \left[e^{-\alpha n} \sum_{i=0}^{\infty} \binom{-L-n}{i} e^{-\alpha i} \left(\frac{1}{G} - 1\right)^i \right]. \tag{51}$$

Performing summation with respect to i, we find

$$2P_{er} = 1 + \sum_{k=0}^{N} \frac{\alpha^k}{k!} (-1)^k \frac{d^k}{d\alpha^k} \left\{ [G - e^{-\alpha}(G-1)]^{-L} \times \right.$$

$$\left. \times \sum_{n=0}^{\infty} [Ge^{\alpha} - G + 1]^{-n} [P_{c+n}(n) - P_n(n)] \right. \tag{52}$$

or, performing the replacement

$$G(e^{\alpha} - 1) = e^{\beta} - 1;$$

$$\beta = \ln [Ge^{\alpha} - G + 1], \tag{53}$$

$$2P_{er} = 1 +$$

$$+ \sum_{k=0}^{N} \frac{\alpha^k}{k!} (-1)^k \frac{d^k}{d\alpha^k} \left\{ e^{[\alpha - \beta(\alpha)] L} \sum_{n=0}^{\infty} e^{-\beta n} [P_{s+n}(n) - P_n(n)] \right\}. \tag{54}$$

For further simplification, it is convenient to introduce the generating functions of the moments of the input distributions

$$F(x) = \sum_n e^{nx} P(n), \tag{55}$$

which have the property that

$$F_{s+n}(x) = F_c(x) F_n(x). \tag{56}$$

Considering this, (54) can be finally rewritten in the form

$$2P_{er} = 1 + \sum_{k=0}^{N} \frac{\alpha^k}{k!} (-1)^k \frac{d^k}{d\alpha^k} \left\{ e^{(\alpha-\beta)L} [F_n(-\beta) F_c(-\beta) - F_n(-\beta)] \right\}. \tag{57}$$

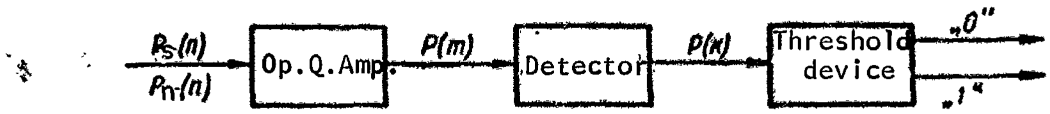

Figure 3. Passage of Signal and Noise Through Binary Information Receiver

Formula (57) determines the probability of an error at the output of the receiver depending on parameters α, G and L, as well as the type of distributions of the input photons of the signal and of the noise. If the amplifier is /243
not used, the probability of error is produced from (57) where G = 1:

$$2P_{er}^0 = 1 + \sum_{k=0}^{N} \frac{\alpha^k}{k!} (-1)^k \frac{d^k}{d\alpha^k} [F_s(-\alpha) F_n(-\alpha) - F_n(-\alpha)]. \tag{58}$$

These last two expressions make it possible to determine the influence of the amplifier on the output error. The presence of the amplifier in the circuit leads to replacement of parameter α with the effective parameter $\beta \approx ln[G(e^{\alpha} - 1)]$ in the generating functions, and to the addition of additive noise with generating function $e^{(\alpha - \beta)L}$.

Optimization of this communications system consists of seeking out the minimum of P_{er}. First of all, with fixed α, G and L, we establish the optimal threshold N_{opt}, which is defined as the integer less than the least positive root of the equation:

$$\frac{d^{N_{opt.}}}{d\alpha^{N_{opt.}}}\left\{e^{(\alpha-\beta)L}\left[F_s(-\beta)F_n(-\beta)-F_n(-\beta)\right]\right\}=0. \qquad (59)$$

The physical sense of this condition can be explained using Figure 4. If the threshold is determined from equation (59), the shaded area on Figure 4, which determines the probability of an error, is minimal (Figure 4a). Since $P_{s+n}(0) < P_n(0)$, the desired threshold value is always definite. After calculating the optimal threshold N_{opt} for a receiver with and without an amplifier and comparing the corresponding minimal errors, we can make a judgment concerning the effectiveness of using the amplifier with the given parameters G and L. In certain particularly simple cases, this task can be partially solved analytically. For example, for the case often seen in the literature when the distribution of photoelectrons of signal and noise is considered Poisson and when there is no amplifier, the optimal threshold is determined precisely. Actually, as was shown in [15], the Poisson distribution of photoelectrons corresponds to a δ-distribution of input photons, so that the generating functions /244
will be equal to:

$$\begin{aligned} F_s(-\alpha) &= e^{-\alpha\bar{n}_s}, \\ F_n(-\alpha) &= e^{-\alpha\bar{n}_n}, \end{aligned} \qquad (60)$$

and the optimal threshold will be

$$N_{opt.} = \frac{\alpha\bar{n}_s}{\ln\left(1+\frac{\bar{n}_s}{\bar{n}_n}\right)}, \qquad (61)$$

where $\bar{n}_s$ and $\bar{n}_n$ are the number of photons in the signal and in the noise at the input. In spite of the fact that the assumption of δ-shaped distribution of photons is an idealization, equation (61) gives a rather good approximation for N_{opt} in certain other more realistic cases as well. Simple calculations show that the selection of the threshold in correspondence with (61) allows a considerable reduction of the average error in certain cases. For example, Flint [21] investigates a system with the following characteristics: $\alpha\bar{n}_n = 5\cdot 10^{-2}$, $\alpha\bar{n}_s = 6\text{-}16$, $N = 5$ and Poisson distribution of photoelectrons. However, it is not difficult to see that the optimal threshold will be $N_{opt} = 1$ where $\alpha\bar{n}_s = 6$ and $N_{opt} = 2$ where $\alpha\bar{n}_s = 16$. The probability of error where $\alpha\bar{n}_s = 6$ will, correspondingly, be as follows: $P_{er.min} = 0.009$; $P_{er}(N = 5) >$ > 0.21. Thus, proper selection of the threshold level allows the mean error to be decreased by more than an order of magnitude.

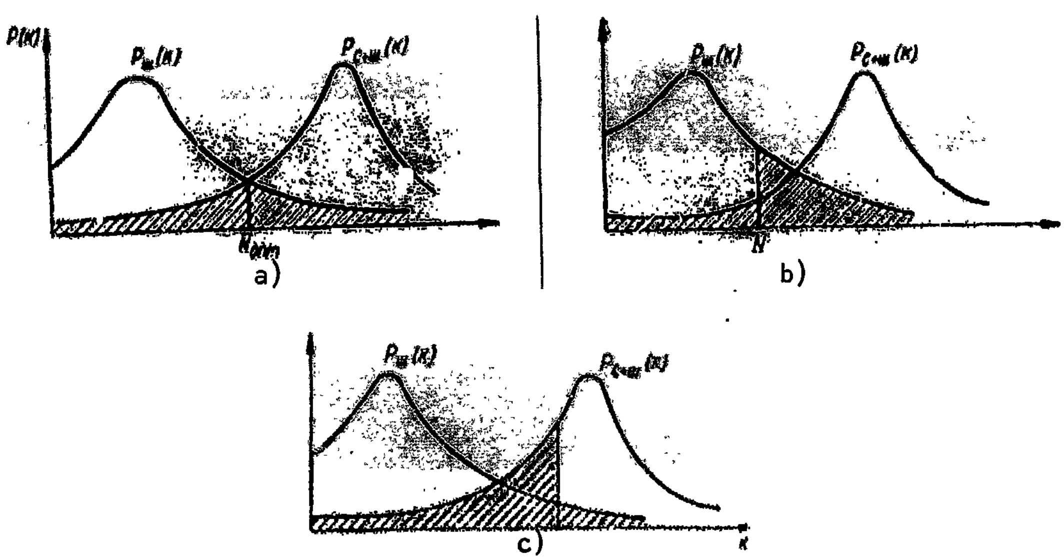

Figure 4. Determination of the Optimal Threshold Level: a, Level optimally; b, c, Level selected nonoptimally

Let us now investigate the influence of an amplifier on the magnitude of error once more, in the assumption of δ-distribution of photons at the input. The general expression for N_{opt} through the values of the parameters cannot be found in this case; however, by representing

$$x_N = (-1)^N \frac{d^N}{d\alpha^N}\left\{e^{(\alpha-\beta)L}\left[e^{-\beta(\bar{n}_s+\bar{n}_n)} - e^{-\beta\bar{n}_n}\right]\right\} =$$
$$= (-1)^N \frac{d^N}{d\alpha^N}\left\{\frac{e^{\alpha L}}{(Ge^{\alpha}+G-1)^{L+\bar{n}_n}}\left[\frac{1}{(Ge^{\alpha}+G-1)^{\bar{n}_s}} - 1\right]\right\}, \qquad (62)$$

we can calculate several of the first values of x_N. Thus,

$$x_0 = \frac{e^{\alpha L}}{(Ge^{\alpha} + G - 1)^{L+\bar{n}_n}} \left[\frac{1}{(Ge^{\alpha} + G - 1)^{\bar{n}_s}} - 1 \right] < 0 \qquad (63)$$

for all $G \geqslant 1$,

$$x_1 = \frac{e^{\alpha L}}{(Ge^{\alpha} + G - 1)^{L+\bar{n}_n+1}} \left[L(G-1) - \bar{n}_n Ge^{\alpha} - \frac{L(G-1) - (\bar{n}_n + \bar{n}_s) Ge^{\alpha}}{(Ge^{\alpha} + G - 1)^{\bar{n}_s}} \right] \times$$

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$$\times \begin{cases} < 0 \text{ where } L < \dfrac{\bar{n}_n Ge^{\alpha}}{G-1} - \dfrac{\bar{n}_s Ge^{\alpha}}{(G-1)\left[(Ge^{\alpha} + G - 1)^{\bar{n}_s} - 1\right]}, \\ > 0 \text{ where } L > \dfrac{\bar{n}_n Ge^{\alpha}}{G-1} - \dfrac{\bar{n}_s Ge^{\alpha}}{(G-1)\left[(Ge^{\alpha} + G - 1)^{\bar{n}_s} - 1\right]} \end{cases} \qquad (64)$$

If the gain G is great enough, $x_1 > 0$ where $L > n_n e^{\alpha}$, i.e. when the internal noise of the amplifier exceeds the noise at the input. In this situation, the optimal threshold level will obviously be $N_{opt} = 0$, i.e. the "0" signal is recorded if there are no photoelectrons at the output of the photo detector, and the "1" will be recorded otherwise. This operating mode is called the binary mode. For a receiver operating in the binary mode, it is easy to calculate the probability of an error at the output:

$$2P_{er} = 1 + x_0 = 1 - \frac{e^{\alpha L}}{(Ge^{\alpha} + G - 1)^{L+\bar{n}_n}} \left[1 - \frac{1}{(Ge^{\alpha} + G - 1)^{\bar{n}_s}} \right], \qquad (65)$$

which reaches a minimum at

$$G = \frac{1}{e^{\alpha} + 1} \left[1 + \frac{\bar{n}_s}{\bar{n}_n + L} \right]^{1/\bar{n}_s}. \qquad (66)$$

It follows from this that where $n_s \geqslant 1$, $G_0 \sim 1$, i.e. a receiver device without an amplifier is optimal. Actually, the usage of an amplifier with a rather high gain and internal noise exceeding the input noise only worsens the operation of the receiver in a binary system. However, if the internal noise of the amplifier is low, it is easy to see that in certain cases (in particular where $\alpha \ll 1$, $n_n \ll 1$) we can expect a certain decrease in the probability of error in comparison to a system without an amplifier. Thus, where $\alpha = 0.05$, $\bar{n}_s = 8$, $\bar{n}_n = 0.05$, we produce for a system without an amplifier $P_{er} \approx 0.335$, and for a system with an amplifier with $G = 20$, $L \ll n_0$, considering only the first two terms we produce $P_{er} < 0.081$, so that the probability of error is decreased by approximately five times with threshold $N = 1$. Actually, for a system with an amplifier with the parameters outlined above, the optimal threshold level $N \sim 30$, as a result of which the error can be even less.

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